OPTIMAL N-IMPULSE TRANSFER BETWEEN COPLANAR ORBITS

Roger A. Broucke*  
Antonio F. B. A. Prado**

In the present research we consider the problem of optimal (minimum \( \Delta V \)) time-free N-impulse transfers between any two coplanar orbits, for the cases where \( N \leq 4 \). For the two-impulse maneuver, we develop optimality conditions that lead to a non-linear system of three equations and three unknowns. The solution of this system gives us the bi-impulse transfer with minimum total \( \Delta V \). For the three-impulse maneuver, we develop a new approach. We use two elliptic transfer orbits that are connected by a negligible impulse applied at an infinite distance from the main body. It is an extension of the bi-elliptic transfer, where the two orbits involved in the transfer are not co-axial. We study in detail and show regions of optimality for the most trivial cases of transfers: between two circular orbits; one circular and one elliptic orbit; two elliptic co-axial orbits. We complete the paper by developing a scheme to reduce the total \( \Delta V \) for some of those maneuvers, by adding a second impulse at infinity, and making it a four-impulse maneuver.

INTRODUCTION

This paper studies the problem of time-free minimum \( \Delta V \) transfers between any two elliptical coplanar orbits. The problem of optimal transfers (in the sense of minimum fuel consumption) between two Kepleriancoplanar orbits have been under investigation for a long time. In particular, many papers solve this problem for an impulsive thrust system with a fixed number of impulses. The literature is full of solutions for particular cases, like the Hohmann\(^1\) and the Hoelker-Silber\(^2\) transfers between two circular orbits and its variants for ellipses in particular positions.

In this paper we derive the equations that give us the optimal solution for a transfer between any two elliptic orbits with a fixed number \( N \) of impulses (\( N \leq 4 \)). For the two-impulse transfer case we follow the theory originally developed by Lawden\(^3,4\). The new aspect of our formulation is that we introduce a new set of variables that allows us to reduce his eleven equations in eleven unknowns to the minimum possible set of three equations in three unknowns. For the case of three and four impulses, new schemes are developed to solve the problem, considering only transfers that go to infinity during the transfer. It is known for more than 30 years now (Ting\(^5\); Moyer\(^6\)) that a maneuver with more than two impulses has to go to infinity to be optimal. We also present numerical tests for the equations derived, showing the savings obtained and the regions of optimality for the case of two, three and four impulses.

* Department of Aerospace Engineering and Engineering Mechanics, University of Texas at Austin.  
** University of Texas at Austin and Instituto Nacional de Pesquisas Espaciais (INPE-Brazil).
REVISION OF THE LITERATURE

R. H. Goddard\(^7\) was one of the first researchers to work on the problem of optimal transfers of a spacecraft between two points. He proposed optimal approximate solutions for the problem of sending a rocket to high altitudes with minimum fuel consumption.

After him comes the very important work done by Hohmann\(^1\). He solved the problem of minimum \(\Delta V\) transfers between two circular coplanar orbits. His results are largely used nowadays as a first approximation of more complex models and it was considered the final solution of this problem until 1959. A detailed study of this transfer can be found in Marec\(^8\) and an analytical proof of its optimality can be found in Barrar\(^9\).

The Hohmann transfer would be generalized to the elliptic case (transfer between two coaxial elliptic orbits) by Marchal\(^10\). Smith\(^11\) shows results for some other special cases, like coaxial and quasi-coaxial elliptic orbits, circular-elliptic orbits, two quasi-circular orbits. A numerical scheme to solve the transfer between two generic coplanar elliptic orbits is presented by Bender\(^12\).

Hohmann type transfers between non-coplanar orbits are discussed in several papers, like McCue\(^13\), that study a transfer between two elliptic inclined orbits including the possibility of rendezvous; or Eckel and Vinh\(^14\) that solve the same problem with time or fuel fixed.

Another line of research studies the effects of the reality of finite thrust in the results obtained from the impulsive model. Zee\(^15\) obtained analytical expressions for the extra fuel consumed to reach the same transfer and for the errors in the orbital elements and energy for a nominal maneuver (a real maneuver that uses the impulses calculated with the impulsive model).

More recently, the literature studied the problem of a two-impulse transfer where the magnitude of the two impulses are fixed, like in Jin and Melton\(^16\); Jezewski and Mittleman\(^17\).

The three-impulse concept was introduced in the literature by Hoelker and Silber\(^2\). They showed that a bi-elliptical transfer between two circular orbits has a lower \(\Delta V\) than the Hohmann transfer, for some combinations of initial and final orbits. After that, Ting\(^5\) showed that the use of more than three impulses does not lower the \(\Delta V\), for impulsive maneuvers. Roth\(^18\) obtained the minimum \(\Delta V\) solution for a bi-elliptical transfer between two inclined orbits.

Following the idea of more than two impulses, we have the work done by Prussing\(^19\) that admits two or three impulses; Prussing\(^20\) that admits four impulses; Eckel\(^21\) that admits \(N\) impulses.

Another line of research that comes from the Hohmann transfer is the study of multi-revolutions transfer with \(N\) impulses applied during \(N\) successive passages by the apses. Spencer, Glickman and Bercaw\(^22\) shows equations and pictures to obtain the \(\Delta V\) required for this transfer, as a function of the number of revolutions allowed for the transfer. After that, Redding\(^23\) and Matogawa\(^24\) would extend this concept of multi-revolution transfer to the non-impulsive case, by applying finite thrust around the apses.

Some other researchers worked on methods where the number of impulses was a free parameter, and not a value fixed in advance. It is the case of the papers made by Lion and Handelsman\(^25\), Jezewski and Rosendaal\(^26\), Gross and Prussing\(^27\), Eckel\(^28\) and Prussing and Chiu\(^29\). Most of the research done in this particular case is based on the "Primer-Vector" theory developed by Lawden\(^30,31\)
THE BI-IMPULSIVE TRANSFER

Suppose that we have a spacecraft in a Keplerian orbit $O_0$. We desire to transfer this spacecraft to a final Keplerian orbit $O_2$, coplanar with $O_0$. Figure 1 shows a sketch of the transfer and defines some of the variables used. At the point $P_1(r_1, \theta_1)$, we apply an impulse with magnitude $\Delta V_1$ that has an angle $\phi_1$ with the local transverse direction. The transfer orbit crosses the final orbit at the point $P_2(r_2, \theta_2)$, where we apply an impulse with magnitude $\Delta V_2$ making an angle $\phi_2$ with the local transverse direction.

![Figure 1 - Geometry of the Transfer for a Bi-Impulsive Maneuver.](image)

Using basic equations from the two-body celestial mechanics, we can write an analytical expression for the total $\Delta V (=\Delta V_1 + \Delta V_2)$ required for this maneuver. To specify each of the three orbits involved in the problem we use the elements $D$, $h$ and $k$, defined by the equations:

$$D = \frac{\mu}{C}; \quad k = e\cos(\omega); \quad h = e\sin(\omega)$$ (1)

where $\mu$ is the gravitational parameter of the central body; $C$ is the angular momentum of the orbit, $e$ is the eccentricity and $\omega$ is the argument of the periapse. We also use the subscripts "0" for the initial orbit, "1" for the transfer orbit and "2" for the final orbit. In those variables, the expressions for the radial (subscript r) and transverse (subscript t) components of the two impulses are:
\[ \Delta V_{r_1} = (D_1 k_1 - D_0 k_0) \sin(\theta_1) - (D_1 h_1 - D_0 h_0) \cos(\theta_1) \] (2)
\[ \Delta V_{r_1} = D_1 - D_0 + (D_1 k_1 - D_0 k_0) \cos(\theta_1) + (D_1 h_1 - D_0 h_0) \sin(\theta_1) \] (3)
\[ \Delta V_{r_2} = (D_2 k_2 - D_1 k_1) \sin(\theta_2) - (D_2 h_2 - D_1 h_1) \cos(\theta_2) \] (4)
\[ \Delta V_{r_2} = D_2 - D_1 + (D_2 k_2 - D_1 k_1) \cos(\theta_2) + (D_2 h_2 - D_1 h_1) \sin(\theta_2) \] (5)

Our problem is to find the transfer orbit that minimizes the total \( \Delta V \), and satisfies the two following constraints equations, expressing the fact that the orbits intersect:

\[ g_1 = D_0^2 (1 + k_0 \cos(\theta_1) + h_0 \sin(\theta_1)) - D_1^2 (1 + k_1 \cos(\theta_1) + h_1 \sin(\theta_1)) = 0 \] (6)
\[ g_2 = D_2^2 (1 + k_2 \cos(\theta_2) + h_2 \sin(\theta_2)) - D_1^2 (1 + k_1 \cos(\theta_2) + h_1 \sin(\theta_2)) = 0 \] (7)

Using optimal control language, our now problem is to minimize \( \Delta V = \sqrt{\Delta V_{r_1}^2 + \Delta V_{r_2}^2 + \Delta V_{r_1}^2} \) subject to the conditions (6) and (7). We can use the constraints (6) and (7) to solve this system for two of our variables, making the equation for the \( \Delta V \) a function of only three independent variables. After algebraic manipulations we can find the equations:

\[ k_i = \text{Csc}(\theta_i - \theta_j) \left[ \left( \frac{D_i'}{D_i} \right) (1 + k_i \cos(\theta_j) + h_i \sin(\theta_j)) - 1 \right] \sin(\theta_j) - \left( \frac{D_i'}{D_i} \right) (1 + k_1 \cos(\theta_i) + h_1 \sin(\theta_i)) - 1 \sin(\theta_i) \] (8)
\[ h_i = \text{Csc}(\theta_i - \theta_j) \left[ \left( \frac{D_i'}{D_i} \right) (1 + k_i \cos(\theta_j) + h_i \sin(\theta_j)) - 1 \right] \cos(\theta_j) - \left( \frac{D_i'}{D_i} \right) (1 + k_1 \cos(\theta_i) + h_1 \sin(\theta_i)) - 1 \cos(\theta_i) \] (9)

Now that our \( \Delta V \) is a function of only three variables (\( D_1, \theta_1 \) and \( \theta_2 \)), we can use elementary calculus to find its minimum. From the definition of \( \Delta V \) we can write:

\[ \frac{\partial (\Delta V)}{\partial \alpha_m} = 0 = \frac{1}{\Delta V} \left[ \Delta V_{r_1} \frac{\partial (\Delta V_{r_1})}{\partial \alpha_m} + \Delta V_{r_1} \frac{\partial (\Delta V_{r_2})}{\partial \alpha_m} \right] + \frac{1}{\Delta V} \left[ \Delta V_{r_2} \frac{\partial (\Delta V_{r_2})}{\partial \alpha_m} + \Delta V_{r_2} \frac{\partial (\Delta V_{r_2})}{\partial \alpha_m} \right] \] (10)

where \( \alpha_1 = D_1, \alpha_2 = \theta_1, \alpha_3 = \theta_2 \).

We can apply the chain rule for derivatives to obtain expressions for the partials involved in equation (10). A general expression for them is:

\[ \frac{\partial (\Delta V_{r_j})}{\partial \alpha_m} = \frac{\partial (\Delta V_{r_j})}{\partial \alpha_m} \bigg|_{\text{Direct}} + \frac{\partial (\Delta V_{r_j})}{\partial \alpha_m} \frac{\partial k_1}{\partial \alpha_m} + \frac{\partial (\Delta V_{r_j})}{\partial \alpha_m} \frac{\partial h_1}{\partial \alpha_m} \] (11)

where \( i = r, t; j = 1, 2 \) and the word "Direct" stands for the part of the derivative that comes from the explicit dependence of \( \Delta V_{r_j} \) in the variable \( \alpha_m \). The expressions for \( \frac{\partial (\Delta V_{r_j})}{\partial k_1} \) and \( \frac{\partial (\Delta V_{r_j})}{\partial h_1} \) can be obtained from equations (2) to (5) and the expressions for \( \frac{\partial k_1}{\partial \alpha_m} \) and \( \frac{\partial h_1}{\partial \alpha_m} \) can be obtained from the equations (8) and (9).

Those equations are shown in Appendix A and numerical results are shown later in this paper.
The 180°-Transfer Theory

From equations (8) and (9) we can see the existence of singularities in our equations. It happens for the transfers where $\theta_1 - \theta_2 = (2m+1)\pi$, where $m$ is any integer. This particular case includes the very important Hohmann-class family. It occurs when the transfer is between two circular orbits, between a circular and an elliptic orbit or between two elliptic co-axial orbits (ellipses with the semi-major axis in the same direction). From the initial data (initial and final orbits) we know in advance if our problem is one of those cases and we can solve it by trivial means, without going through our theory. But, if desired, we can easily adapt our theory to solve this class of transfers, by solving the constraints (6) and (7) for the variables $D_1$ and $k_1$ and following the same guidelines after that, with $h_1, \theta_1$ and $\theta_2$ as our new set of independent variables. We do not show a full development of this 180°-transfer theory here, because equations (8) and (9) generate less algebraic work in the continuation of the theory. This is due to the fact that equations (6) and (7) are symmetric in the variables $h_1$ and $k_1$. We give here only the expressions for $D_1$ and $k_1$. They are:

$$D_1 = \sqrt{\frac{D_2^2 \cos(\theta_1)\left[1 + h_1\sin(\theta_1)\right] - D_0^2 \cos(\theta_2)\left[1 + h_0\sin(\theta_1)\right] + [D_2^3 k_2 - D_0^3 k_0] \cos(\theta_1) \cos(\theta_2)}{\cos(\theta_1) - \cos(\theta_2) - h_1 \sin(\theta_1 - \theta_2)}}\quad(12)$$

$$k_1 = \frac{D_2^2 - D_0^2 + D_0^3 k_0 \cos(\theta_1)\left[1 + h_1\sin(\theta_1)\right] - D_2^3 k_2 \cos(\theta_2)\left[1 + h_0\sin(\theta_1)\right] + [D_0^3 h_0 - D_2^3 h_1] \sin(\theta_1) +}{D_2^2 \cos(\theta_1)\left[1 + h_2\sin(\theta_2)\right] - D_0^2 \cos(\theta_2)\left[1 + h_0\sin(\theta_1)\right] - [D_0^3 k_0 - D_2^3 k_2] \cos(\theta_1) \cos(\theta_2)} + \frac{[D_0^3 h_0 - D_2^3 h_1] \sin(\theta_1) + [D_0^3 h_0 - D_2^3 h_1] \sin(\theta_2) \sin(\theta_1) +}{D_2^2 \cos(\theta_1)\left[1 + h_2\sin(\theta_2)\right] - D_0^2 \cos(\theta_2)\left[1 + h_0\sin(\theta_1)\right] - [D_0^3 k_0 - D_2^3 k_2] \cos(\theta_1) \cos(\theta_2)}\quad(13)$$

We can see that those equations have no singularities, even in the case $\theta_1 - \theta_2 = (2m+1)\pi$. We check the validity of this theoretical variant by verifying that the equations (10) are satisfied in all the trivial cases that we cited before (circular-circular, circular-ellipse, ellipse-ellipse co-axial).

THREE-IMPULSE TRANSFERS THROUGH INFINITY

As we said before, there is no local minimum in the $\Delta V$ for a time-free maneuver with three or more impulses that does not go to infinity. The details and the proof of this assumption can be found in Ting\textsuperscript{5}. So, we now study in details those maneuvers in this paper. We compare different possibilities of three-impulse transfers against each other and against the optimal two-impulse maneuver, for the most trivial cases. For the more generic case of two elliptic non-aligned orbits we develop a new approach and a new set of equations to solve this problem. Let us proceed, studying case by case.

Transfers between two circular orbits

After the discovery that the Hohmann transfer is the optimal bi-impulsive transfer between two coplanar circular orbits\textsuperscript{1}, many other important steps were made by several researchers. Hoelker and Silber\textsuperscript{2} showed that a bi-elliptical three-impulsive transfer would give us a lower $\Delta V$, when the ratio of the radius of the two orbits involved is greater than 11.93875. This transfer is accomplished in three steps: i) We apply the first impulse to send the spacecraft from its initial orbit to a first elliptic transfer orbit with apoapse distance $R_A$ greater than the radius of the final orbit; ii) At this point, we apply a second impulse, with near-zero $\Delta V$, to transfer the spacecraft to a second elliptic transfer orbit that put the spacecraft in a path that crosses the final orbit; iii) We apply the last impulse when the spacecraft crosses its final orbit and this impulse makes the spacecraft stay in that final orbit. The savings provided by this transfer increases when we increase the distance $R_A$ of the point where the intermediate impulse is applied, at an expense of an increase in the time.
for the transfer. The maximum for the savings in $\Delta V$ is given by the bi-parabolic transfer, where both transfer orbits are parabolic and the second impulse is applied at infinity and it has a zero-magnitude. In this paper we use the variables that we defined previously to obtain equations for the savings that a transfer through infinity can give us over the standard bi-impulse maneuvers and the expression for the extra time required to complete the transfer. The savings in Delta-V ($\Delta V_{SAV}$), defined as the difference in $\Delta V$ between the three-impulsive transfer ($\Delta V_{TRI}$) and the standard Hohmann transfer ($\Delta V_{HOH}$) is given by:

$$\Delta V_{SAV} = \Delta V_{TRI} - \Delta V_{HOH}$$

(14)

where,

$$\Delta V_{TRI} = D_0 \left\{ -1 - \frac{1}{r} + \sqrt{2 \left( \frac{1}{r^2} + \frac{1}{D_0^2 R_A^2} \right)} + \frac{\sqrt{2 (D_0^2 R_A^2 - 1)}}{D_0 \sqrt{R_A (1 + D_0^2 R_A^2)}} \right\};$$

(15)

$$\Delta V_{HOH} = D_0 \left\{ -1 + \frac{1}{r} + \frac{\sqrt{2 (r^2 - 1)}}{r \sqrt{1 + r^2}} \right\};$$

(16)

$$r = \frac{D_0}{D_2} = \sqrt{\frac{a_2}{a_0}};$$

(17)

$R_A$ = the distance between the main body and the spacecraft in the moment that the second impulse is applied;
a_0, a_2 = the semi-major axis of the initial and the final orbits, respectively.

If we consider only the limit case $R_A \to \infty$, the $\Delta V_{SAV}$ goes to:

$$\Delta V_{SAV} (R_A \to \infty) = D_0 \left( \sqrt{2 - \frac{2}{r}} + \frac{\sqrt{2}}{r} + \frac{\sqrt{2 (1 - r^2)}}{r \sqrt{1 + r^2}} \right)$$

(18)

The extra time required for these transfers ($\Delta T_{EXTRA}$), that means $T_{TRI}$ (time for the three impulses transfer) - $T_{HOH}$ (time for the Hohmann transfer) is given by:

$$\Delta T_{EXTRA} = T_{TRI} - T_{HOH}$$

(19)

where:

$$T_{TRI} = \pi \left( \frac{r^2 + D_0^2 R_A}{2D_0^2} \right)^{\frac{3}{2}} + \left( \frac{1 + D_0^2 R_A}{2D_0^2} \right)^{\frac{3}{2}}$$

(20)

$$T_{HOH} = \pi \left( \frac{1 + r^2}{2D_0^2} \right)^{\frac{3}{2}}$$

(21)

We know that not all the cases (combination of initial and final orbit) can give us savings by using a three-impulse maneuver. This possibility depends on the value of $r$. The first value for $r$ that allows us to obtain a positive saving is:
\[
\lim_{r \to \infty} \Delta V_{SAV} = D_0 \left( \frac{\sqrt{2} - \frac{2}{r} + \frac{\sqrt{2}}{r} - \frac{\sqrt{2}(1-r^2)}{r\sqrt{1+r^2}}} \right) = 0 \Rightarrow r = 3.45525 \Rightarrow r^2 = \frac{a_2}{a_0} = 11.93875
\]  
(22)

that is in agreement with the literature\(^8\).

The \(\Delta V_{SAV}\) against the TIME for the transfer (\(\Delta T_{EXTRA}\)) is shown in Figure 2 for several values of \(r\) and assuming \(D_0 = 1\), that is only a scale for distance and does not make any restriction in the problem.

![Figure 2 - \(\Delta V\) Saved by Using Three Impulses Against Time as a Function of \(r\).](image)

The units are chosen such that the gravitational parameter and the quantity \(D_0\) are both units, which corresponds to the canonical system of units for this case.

We can see easily that the savings increase when we allow more time for the transfer, and that all the savings tend to a certain limit (the bi-parabolic transfer) when the time is large enough.

**Transfers between one circular and one elliptic orbit**

This case is a little more complex, and it allows us to choose between two Hohmann-type transfers and two three-impulse transfers. In both cases we have the transfer orbit that inserts the spacecraft at the apoapse (H1 and TRI2) or at the periapse (H2 and TRI1) of the final orbit.

Marchal\(^3^2\) shows that between the two choices available for the Hohmann-type transfer, the one with minimum \(\Delta V\) is the one that uses the apoapse of the final orbit, in this case H1. It is easy to see that this H1 requires less \(\Delta V\), but it requires more time for the transfer, because the semi-major axis of the transfer orbit is larger.

To find out which one of the three-impulse transfer has a lower \(\Delta V\), we calculate the difference in \(\Delta V\) for both cases, in the limiting situation where the second \(\Delta V\) is applied at an infinity distance. The result, after algebraic manipulations, is:

\[
\Delta V_{TRI1-TRI2} = \lim_{r \to \infty} (\Delta V_{TRI1} - \Delta V_{TRI2}) = D_0 \left( k_2 + \sqrt{2(1-k_2)} - \sqrt{2(1+k_2)} \right)
\]  
(23)
If we remember that $D_2 > 0$, we can plot this expression against $k_2$, to get the following conclusions:

$$\Delta V_{\text{TRI}1-\text{TRI}2} > 0, \text{ if } k_2 > 0$$
$$\Delta V_{\text{TRI}1-\text{TRI}2} < 0, \text{ if } k_2 < 0$$

Those conclusions can be transformed in words by saying that: "the three-impulsive transfer that inserts the spacecraft at the periapses of the second orbit is always better, in terms of having a smaller $\Delta V$ required for the transfer". Another advantage of TRI1 is that it requires a smaller time for the transfer, for a fixed value of $R_A$.

After deciding the best Hohmann-type transfer and the best three-impulse transfer, we can derive the equations for the savings in $\Delta V$ and the extra-time required for the transfer, as we did in the previous section, to compare the best two-impulse transfer against the best three-impulse transfer. The equations are:

$$\Delta V_{\text{SAV}} = \Delta V_{\text{TRI}} - \Delta V_{\text{HOH}}$$

where,

$$\Delta V_{\text{HOH}} = D_2 - D_0 - D_2 k_2 + \sqrt{2 \left( D_0^2 - D_2^2 (1 - k_2) \right)} \over \sqrt{D_0^2 + D_2^2 (1 - k_2)}$$

is the $\Delta V$ for the best Hohmann transfer and

$$\Delta V_{\text{TRI}} = -D_0 - D_2 - D_2 k_2 + \sqrt{2 \left( D_2^2 (1 + k_2) + \frac{1}{R_A} \right)}$$

is the $\Delta V$ for the best three-impulse transfer.

For the extra time required we have:

$$\Delta T_{\text{EXTRA}} = T_{\text{TRI}} - T_{\text{HOH}}$$

where,

$$T_{\text{TRI}} = \pi \left( \sqrt{\left( \frac{1}{2D_0^2} + \frac{R_A}{2} \right)^3} + \sqrt{\left( \frac{1}{2D_2^2 (1 + k_2)} + \frac{R_A}{2} \right)^3} \right)$$

is the time required to complete the three-impulse transfer and

$$T_{\text{HOH}} = \pi \sqrt{\left( \frac{1}{2D_0^2} + \frac{1}{2D_2^2 (1 - k_2)} \right)^3}$$

is the time required to complete the Hohmann transfer.
Figure 3 shows $\Delta V_{SAV} \times \Delta T_{EXTRA}$ for the case $D_0 = 1$ as a function of $D_2$ and $k_2$ and the same $\Delta V_{SAV} \times \Delta T_{EXTRA}$ as a function of the Keplerian elements. Figure 4 shows the level-curves for $\Delta V_{SAV}$ to show the region of the two-dimensional space $D_2$-$k_2$ that allows savings in $\Delta V$ by using a transfer through infinity. We can see the existence of three regions: Region A is the forbidden region of our equations, where the orbits intersect and our equations have to be modified; Region B is the region where the savings are negative and the two-impulse maneuver has a lower $\Delta V$; Region C is the region where the savings are positive and the three-impulse maneuver has a lower $\Delta V$. We show these results in our special set of variables and in the more standard Keplerian elements semi-major axis and eccentricity.

$\Delta V$ Saved Against Time for the Transfer for $D_0 = 1$, $D_2 = 0.2$ and $-0.9 \leq k_2 \leq 0.9$.

$\Delta V$ Saved Against Time for the Transfer for $D_0 = 1$, $k_2 = -0.3$ and $0.1 \leq D_2 \leq 0.3$.

$\Delta V$ Saved Against Time for the Transfer for $a_0 = 1$, $e_2 = 0.5$ and $25 \leq a_2 \leq 100$.

$\Delta V$ Saved Against Time for the Transfer for $a_0 = 1$, $a_2 = 50$ and $0.0 \leq e_2 \leq 0.6$.

Figure 3 - $\Delta V$ Saved by Using Three Impulses Against Time for the Transfer.
Figure 4 - Level Curves for $\Delta V_{SAV}$. 
Transfers between two co-axial elliptic orbits

For this case we have several different possibilities. First of all, the orbits may have their periapse in the same direction (we call it aligned orbits) or in opposite direction (we call it opposite orbits). We also have two possibilities for a Hohmann-type and for a three-impulse type transfer orbits: we can start the transfer at the periapse or at the apoapse of the initial orbit. We name all those cases in the following way:

H1: A Hohmann-type transfer starting at the periapse of the initial orbit;
H2: A Hohmann-type transfer starting at the apoapse of the initial orbit;
TRI1: A three-impulse transfer orbit starting at the periapse of the initial orbit;
TRI2: A three-impulse transfer orbit starting at the apoapse of the initial orbit.

The literature has a rule to choose the minimum $\Delta V$ transfer between the two Hohmann-type transfers: the best one is the one that uses the most distant apoapse.

In this paper we address the problem of comparing the two three-impulse transfers between themselves and the best three-impulse against the best Hohmann-type transfers.

For the comparison of the two three-impulse transfers we derived an expression for the difference in $\Delta V$ between themselves. For this comparison we assume the limit case for both transfers, which means that both three-impulse transfers go to infinity in the intermediate step. The final expression, after algebraic manipulation, is:

$$
\Delta V_{\text{TRI1}} - \Delta V_{\text{TRI2}} = D_2 \sqrt{2(1+k_2)} - D_2 \sqrt{2(1-k_2)} - 2D_2 k_2 + \sqrt{2(1+k_0)} - \sqrt{2(1-k_0)} - 2k_0
$$

where we define our reference system such that $0 \leq k_0 \leq 1$ and $-1 \leq k_2 \leq 1$, which means that we start to measure angles from the periapse of the initial orbit.

It is important to note that the elements that we use to describe the orbits ($D$, $h$, $k$) are very appropriate for this expression, because the expression is unchanged for aligned and opposed orbits.

From this expression, we get the following conclusions:

* For $0 \leq k_2 \leq 1$ (aligned orbits) $\Delta V_{\text{TRI1}} - \Delta V_{\text{TRI2}}$ is always negative, what means that the maneuver TRI1 is always better than TRI2.

* For $-1 \leq k_2 \leq 0$ (opposed orbits) $\Delta V_{\text{TRI1}} - \Delta V_{\text{TRI2}}$ can be positive or negative. So, we set $\Delta V_{\text{TRI1}} - \Delta V_{\text{TRI2}} = 0$ and solve it for $D_2$, to find the separation point of optimality between the two orbits. The result is:

$$
D_{2\text{CRI1}} = \frac{\sqrt{2(1+k_0)} - \sqrt{2(1-k_0)} - 2k_0}{\sqrt{2(1-k_2)} - \sqrt{2(1+k_2)} + 2k_2}
$$

and we know that:

If $D_2 = D_{2\text{CRI1}}$, $\Delta V_{\text{TRI1}} = \Delta V_{\text{TRI2}}$

If $D_2 > D_{2\text{CRI1}}$, $\Delta V_{\text{TRI1}} < \Delta V_{\text{TRI2}}$

If $D_2 < D_{2\text{CRI1}}$, $\Delta V_{\text{TRI1}} > \Delta V_{\text{TRI2}}$
Figure 5 shows the surface $\Delta V_{T1-T2} = 0$. This surface divides the 3-D space ($k_0$, $k_2$, $D_2$) in two regions: the one below the surface ($D_2 < D_{2\text{CRI}}$) where TRI2 is better than TRI1 and the one above the surface ($D_2 > D_{2\text{CRI}}$) where TRI1 is better than TRI2.

**Figure 5** - Surface $\Delta V_{T1-T2} = 0$ in the Volume ($k_0$, $k_2$, $D_2$).

We are now ready to follow the same procedure to compare the best transfer of the three-impulse class against the best Hohmann-type transfer.

First of all, let us show the basic equations. We follow the same nomenclature we used in the previous cases. For the $\Delta V$s we have:

$$\Delta V_{SAV} = \Delta V_{TRI} - \Delta V_{HOH}$$  \hspace{1cm} (32)

where

$$\Delta V_{HOH} = D_2 -1 - k_0 - D_2 k_2 + \sqrt{2[1 + D_2^2(1-k_2)+k_0]} \left[ \frac{2(1 + k_0)}{1 + k_0 + D_2^2(1-k_2)} - 1 \right]$$  \hspace{1cm} (33)

for aligned orbits, or

$$\Delta V_{HOH} = D_2 -1 + k_0 + D_2 k_2 - \sqrt{2[1 + D_2^2(1+k_2)-k_0]} \left[ \frac{2(-1 + k_0)}{1 - k_0 + D_2^2(1+k_2)} + 1 \right]$$  \hspace{1cm} (34)

for opposed orbits;

$$\Delta V_{TRI} = -1 - D_2 (1+k_2) - k_0 + \sqrt{2 \left[ 1 + k_0 + \frac{1}{R_A} \right] \left[ \frac{(1+k_0)R_A - 1}{(1+k_0)R_A + 1} \right] + \sqrt{2 \left[ 1 + D_2^2 R_A (1+k_2) \right] \frac{R_A}{R_A}}}$$  \hspace{1cm} (35)
\[ \Delta V_{\text{TRI}2} = -1 - D_2(1 - k_2) + k_0 - \sqrt{2\left(1-k_0 + \frac{1}{R_A}\right)\left(k_0 - 1\right)R_A + 1} + \sqrt{2\left[1 + D_2^2R_A(1-k_2)\right] R_A} \]  

(36)

Those equations are transformed, in the limit case \( R_A, t \to \infty \), in the expressions:

\[ \Delta V_{\text{TRI}1} = \sqrt{2(1+k_0)} - 1 - D_2\left(1 + k_2 - \sqrt{2(1+k_2)}\right) - k_0; \]  

(37)

\[ \Delta V_{\text{TRI}2} = \sqrt{2(1-k_0)} - 1 - D_2\left(1 - k_2 - \sqrt{2(1-k_2)}\right) + k_0. \]  

(38)

For the extra time required for the transfer, we have:

\[ \Delta T_{\text{EXTRA}} = T_{\text{TRI}} - T_{\text{HOH}}, \]  

(39)

\[ T_{\text{HOH}} = \pi \left[ \frac{1}{2(1+k_0)} + \frac{1}{2D_2^2(1-k_2)} \right] \]  

for aligned orbits;  

(40)

\[ T_{\text{HOH}} = \pi \left[ \frac{1}{2(1-k_0)} + \frac{1}{2D_2^2(1+k_2)} \right] \]  

for opposed orbits;  

(41)

\[ T_{\text{TRI}1} = \pi \left[ \left(1 + \frac{R_A}{2}\right) \right] + \left[ \frac{1}{2D_2^2(1+k_2)} + \frac{R_A}{2} \right] \]  

(42)

\[ T_{\text{TRI}2} = \pi \left[ \left(1 + \frac{R_A}{2}\right) \right] + \left[ \frac{1}{2D_2^2(1-k_2)} + \frac{R_A}{2} \right] \]  

(43)

We now repeat the process of defining a surface \( \Delta V_{\text{SAV}} = 0 \) in the 3-D space \((k_0, k_2, D_2)\) to divide the volume in two parts: the one below the surface, where the three-impulse maneuver is better and the one above the surface, where the Hohmann-type is better. Figure 6 shows this division and Figure 7 shows the \( \Delta V_{\text{SAV}} \times \Delta T_{\text{EXTRA}} \) keeping two parameters fixed and varying only the third one. Remember that we are always using the limit case \( R_A, t \to \infty \) for the calculations.
Transfers between any two coplanar elliptic orbits

In this section we derive a set of equations to find an optimal three-impulse transfer through infinity between two generic coplanar elliptic orbits. We define the coordinate system such that the periapse of the initial orbit lies in the positive region of the x-axis. We also choose the semi-major axis of the initial orbit as our unit of distance. In that way, we can say that the Keplerian elements \((a,e,\omega)\) of these orbits are: \((1,e,0)\) for the initial orbit and \((a_2,e_2,\omega_2)\) for the final orbit. We use Keplerian elements instead of the new elements that we defined in this paper for the previous cases because this choice gives us equations in a simpler form. The maneuver will use two intermediate elliptic (quasi-parabolic) orbits with eccentricity assumed to be one and semi-major axis infinity. The maneuver has three basic steps:

**Figure 6** - Surface \(\Delta V_{SAV} = 0\) in the Volume \((k_0, k_2, D_2)\).

**Figure 7** - \(\Delta V\) Saved by Using Three Impulses Against Time for the Transfer.

\[ D_0 = 1, k_0 = 0.2; k_2 = 0.4; 0.10 \leq D_2 \leq 0.25. \]

\[ D_0 = 1, k_0 = 0.2; D_2 = 0.2; -0.9 \leq k_2 \leq 0.9. \]

\[ D_0 = 1, k_2 = 0.4; D_2 = 0.2; 0.0 \leq k_2 \leq 0.9. \]
1) We apply the first impulse when the spacecraft has true anomaly $\theta_0$ in the initial orbit. This impulse makes the spacecraft go to the first intermediate quasi-parabolic transfer orbit with argument of periapse $\omega$.

2) From this orbit, when the spacecraft is at an infinity distance from the main body (apoapse of the quasi-parabolic orbit), we apply the second impulse (with zero magnitude) to transfer the spacecraft to a second quasi-parabolic orbit with the same argument of periapse $\omega$. In this orbit the spacecraft will cross the final orbit in a point that makes an angle $\theta_1$ with a reference line, adopted as the apsidal line of the initial orbit.

3) At this point, we apply the last impulse to capture the spacecraft into its final orbit.

Our task now is to find an expression for the total $\Delta V$ required for this transfer ($= \Delta V_1 + \Delta V_2$) as a function of the three variables $\theta_0$, $\theta_1$, $\omega$. We do that by combining some basic equations from Celestial Mechanics in the way that we show below.

We calculate the distance from the main body ($r_0$), the radial velocity ($V_{r_0}$) and the transverse velocity ($V_{t_0}$) for the spacecraft when in the initial orbit, just before the first impulse. Then we calculate the semi-lactus rectum ($p_1$), the radial velocity ($V_{r_1}$) and the transverse velocity ($V_{t_1}$) for the spacecraft when in the first transfer orbit, just after the first impulse. The equations are:

\[
\begin{align*}
    r_0 &= \frac{1 - e^2}{1 + e\cos(\theta_0)} \\
    V_{r_0} &= \frac{e\sin(\theta_0)}{\sqrt{1 - e^2}} \\
    V_{t_0} &= \frac{1 + e\cos(\theta_0)}{\sqrt{1 - e^2}} \\
    p_1 &= r_0 \left[ 1 + \cos(\theta_0 - \omega) \right] \\
    V_{r_1} &= \frac{e_2 \sin(\theta_1 - \omega_2)}{\sqrt{a_2 \left( 1 - e_2^2 \right)}} \\
    V_{t_1} &= \frac{1 + e_2 \cos(\theta_1 - \omega_2)}{\sqrt{a_2 \left( 1 - e_2^2 \right)}}
\end{align*}
\]

And then, we can make $\Delta V_1 = \sqrt{(V_{r_{10}} - V_{r_0})^2 + (V_{t_{10}} - V_{t_0})^2}$.

We repeat this process for the third impulse (remember that the second impulse has magnitude zero), and we calculate the distance from the main body ($r_1$), the radial velocity ($V_{r_1}$) and the transverse velocity ($V_{t_1}$) for the spacecraft when in the final orbit, just after the third impulse. Then we calculate the semi-lactus rectum ($p_2$), the radial velocity ($V_{r_{11}}$) and the transverse velocity ($V_{t_{11}}$) for the spacecraft when in the second transfer orbit, just before the third impulse. The equations are:

\[
\begin{align*}
    r_1 &= \frac{a_2 \left( 1 - e_2^2 \right)}{1 + e_2 \cos(\theta_1 - \omega_2)} \\
    V_{r_1} &= \frac{e_2 \sin(\theta_1 - \omega_2)}{\sqrt{a_2 \left( 1 - e_2^2 \right)}} \\
    V_{t_1} &= \frac{1 + e_2 \cos(\theta_1 - \omega_2)}{\sqrt{a_2 \left( 1 - e_2^2 \right)}} \\
    p_2 &= r_1 \left[ 1 + \cos(\theta_1 - \omega) \right] \\
    V_{r_{11}} &= \frac{\sin(\theta_1 - \omega)}{\sqrt{p_2}} \\
    V_{t_{11}} &= \frac{1 + \cos(\theta_1 - \omega)}{\sqrt{p_2}}
\end{align*}
\]
And, again, we can make $\Delta V_2 = \sqrt{(V_{t1l} - V_{t1})^2 + (V_{t1l} - V_{t1})^2}$, what gives us the total $\Delta V (= \Delta V_1 + \Delta V_2)$.

Then, we can find the analytical derivatives of the $\Delta V$ with respect to the three variables ($\theta_0$, $\theta_1$, $\omega$), set them equal to zero and solve the resulting system of three equations and three unknowns, to get the solution that gives us the minimum for the $\Delta V$. We do not show the equations here to save space, but they are easy to code and fast to run (less than one second in a PC 486/33).

With those equations we can solve some examples to find the minimum two and three-impulse optimum transfers, using the theories that we developed in this paper. Table 1 shows the results. Note that a positive saving means that the three-impulse maneuver requires a smaller $\Delta V$ and a negative saving means that the two-impulse maneuver requires a smaller $\Delta V$.

### Table 1

<table>
<thead>
<tr>
<th>Orbital Elements</th>
<th>Two-Impulse Transfer</th>
<th>Three-Impulse Transfer</th>
<th>Saving</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta V_1$</td>
<td>$\Delta V_2$</td>
<td>$\Delta V_{tot}$</td>
</tr>
<tr>
<td>$e$ 0.3</td>
<td>$e_2$ 0.3</td>
<td>$a_2$ 10</td>
<td>$\omega$ 60</td>
</tr>
<tr>
<td>$e$ 0.3</td>
<td>$e_2$ 0.3</td>
<td>$a_2$ 10</td>
<td>$\omega$ 120</td>
</tr>
<tr>
<td>$e$ 0.3</td>
<td>$e_2$ 0.3</td>
<td>$a_2$ 30</td>
<td>$\omega$ 60</td>
</tr>
<tr>
<td>$e$ 0.3</td>
<td>$e_2$ 0.3</td>
<td>$a_2$ 30</td>
<td>$\omega$ 120</td>
</tr>
<tr>
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<td>$e_2$ 0.6</td>
<td>$a_2$ 10</td>
<td>$\omega$ 60</td>
</tr>
<tr>
<td>$e$ 0.6</td>
<td>$e_2$ 0.6</td>
<td>$a_2$ 10</td>
<td>$\omega$ 120</td>
</tr>
<tr>
<td>$e$ 0.6</td>
<td>$e_2$ 0.6</td>
<td>$a_2$ 30</td>
<td>$\omega$ 60</td>
</tr>
<tr>
<td>$e$ 0.6</td>
<td>$e_2$ 0.6</td>
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<td>$\omega$ 120</td>
</tr>
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<td>$e$ 0.6</td>
<td>$e_2$ 0.6</td>
<td>$a_2$ 10</td>
<td>$\omega$ 60</td>
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<td>$\omega$ 120</td>
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<td>$e_2$ 0.6</td>
<td>$a_2$ 30</td>
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<td>$e$ 0.6</td>
<td>$e_2$ 0.6</td>
<td>$a_2$ 30</td>
<td>$\omega$ 60</td>
</tr>
<tr>
<td>$e$ 0.6</td>
<td>$e_2$ 0.6</td>
<td>$a_2$ 30</td>
<td>$\omega$ 120</td>
</tr>
</tbody>
</table>

We can see that we reached positive savings for many of the cases tested. This maneuver can not be used in practical applications in the way it is, due to the infinity time required to complete the transfer. It represents the limit case of a practical maneuver that has the second impulse applied at a finite distance (as long as the time constraint allows us) with a finite (but small) $\Delta V$.

### FOUR-IMPULSE TRANSFER THROUGH INFINITY

For the trivial cases of transfers (circular-circular, circular-ellipse, coaxial ellipses) we agree with Ting\(^5\) that there is no how to decrease the total $\Delta V$ by applying more than three impulses. For the most general case of two generic ellipses, we are able to find a scheme that can decrease the total $\Delta V$ by using a fourth impulse. This four-impulse maneuver is performed in the following four steps:

1) The first impulse is applied when the spacecraft is passing by the periapse of the initial orbit. The magnitude should be the $\Delta V$ required to make the spacecraft achieve quasi-parabolic escape velocity at that point. Assuming a transfer from an orbit with Keplerian elements $a = 1$, $e = e$, $\omega = 0$, we have:
\[
\Delta V_1 = \sqrt{\frac{2}{1-e^1}} - \sqrt{\frac{2}{1-e^2}} - 1 \tag{46}
\]

2) After this first impulse we have to wait until the spacecraft reaches the apoapse of this first quasi-parabolic transfer. Then we apply a second impulse with zero magnitude to circularize the orbit (remember that the radius of this circular orbit is infinity);

3) After this second impulse we wait until the spacecraft increases its true anomaly by an angle \( \omega \). At this point the spacecraft is 180° apart from the periapse of the final orbit. This is the right point to apply a third impulse with zero magnitude, to transfer the satellite from this circular orbit to a parabolic orbit with periapse coincident with the periapse of the final orbit;

4) For the final step, we wait until the spacecraft reaches the periapse of its second quasi-parabolic transfer orbit. At this moment we apply the fourth and final impulse to capture the spacecraft into its final orbit. Assuming a final orbit with orbital elements \( a, e, \omega \) we have:

\[
\Delta V_4 = \sqrt{\frac{2}{a_2(1-e_2^2)}} - \sqrt{\frac{2}{a_2(1-e_2^2)}} - \frac{1}{a_2} \tag{47}
\]

Table 2 shows the \( \Delta V \)s for the same cases we studied before, as well as the saving over the two and three impulses maneuvers.

<table>
<thead>
<tr>
<th>Orbital Elements</th>
<th>Four-Impulse Transfer</th>
<th>Four x Two</th>
<th>Four x Three</th>
</tr>
</thead>
<tbody>
<tr>
<td>e ( e_2 ) ( a_2 ) ( \omega )</td>
<td>( \Delta V_1 ) ( \Delta V_2 ) ( \Delta V_{\text{tot}} )</td>
<td>( \Delta V_{\text{SAV}} )</td>
<td>( \Delta V_{\text{SAV}} )</td>
</tr>
<tr>
<td>0.3 0.3 10 60</td>
<td>0.3275 0.1036 0.4311</td>
<td>0.0175</td>
<td>0.0169</td>
</tr>
<tr>
<td>0.3 0.3 10 120</td>
<td>0.3275 0.1036 0.4311</td>
<td>0.0631</td>
<td>0.0450</td>
</tr>
<tr>
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<td>0.1009</td>
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<td>0.0461</td>
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CONCLUSIONS

We derived a new set of equations to solve the problem of optimal transfers between two generic coplanar elliptic orbits in a non-linear system of three equations and three unknowns.
We also derived a new approach and a new set of equations to solve the problem of optimal transfers between two coplanar elliptic orbits by using three impulses, with the second one applied at infinity. We reduced this problem to the problem of solving a non-linear system of three equations and three unknowns.

We compared both schemes by solving several transfers in both cases.

We also derived a scheme to calculate minimum transfers with four impulses, two of them applied at infinity.

We mapped regions of optimality for the more trivial cases of transfers between circular orbits, circular-to-elliptic orbits and between two coaxial elliptic orbits.

ACKNOWLEDGMENTS

The second author wishes to express his thanks to CAPES (Federal Agency for Post-Graduate Education - Brazil) for contributing to this research by giving him a scholarship.

APPENDIX A - EQUATIONS FOR OPTIMAL BI-IMPULSE TRANSFER

Let us introduce the following nomenclature:

\[ J_i = \frac{\partial (\Delta V_{1i})}{\partial \alpha_i}; \quad K_i = \frac{\partial (\Delta V_{1i})}{\partial \alpha_i}; \quad L_i = \frac{\partial (\Delta V_{1i})}{\partial \alpha_i}; \quad M_i = \frac{\partial (\Delta V_{1i})}{\partial \alpha_i} \]  

(A.1)

Using this nomenclature, we have the following expressions for the quantities involved:

\[ J_1 = -\frac{\text{Csc}(\theta_1 - \theta_2)}{2D_1^2}\left[2(D_1^2 + D_2^2) - 2(D_0^2 + D_1^2)\cos(\theta_1 - \theta_2) + D_0^2k_0\cos(2\theta_1 - \theta_2) + (2D_2^2k_2 - D_0^2k_0)\cos(\theta_2) - ...\right.\]

\[ ... - D_0^2h_0\sin(2\theta_1 - \theta_2) + \left(2D_2^2h_2 - D_0^2h_0\right)\sin(\theta_2) \right] \]

(A.2)

\[ J_2 = \frac{\text{Csc}\left(\theta_1 - \theta_2\right)}{4D_1}\left[4(D_2^2 - D_1^2) + 2(2D_2^2k_2 - D_2D_1k_0 - D_1^2k_0)\cos(\theta_1 - \theta_2) + (D_1^2k_2 + D_2D_1k_0 - 2D_2^2k_2)\cos(\theta_1 - \theta_2) + ...\right.\]

\[ ... + (D_2D_1k_0 - D_1^2k_0)\cos(3\theta_1 - 2\theta_2) + 4(D_2^2 - D_1^2)\cos(\theta_1 - \theta_2) + 2(2D_2^2h_0 - D_2D_1h_0 - D_1^2h_2)\sin(\theta_1) + ...\]

\[ ... + \left(2D_2^2h_2 - D_0^2h_0 - D_0D_1h_0\right)\sin(\theta_1 - 2\theta_2) + (D_0D_1h_0 - D_0^2h_0)\sin(3\theta_1 - 2\theta_2) \right] \]

(A.3)

\[ J_3 = \frac{\text{Csc}\left(\theta_1 - \theta_2\right)}{D_1}\left[D_1^2 - D_0^2\right] + \left(D_2^2 - D_1^2\right)\cos(\theta_1) + (D_2^2 - D_1^2)\cos(\theta_1 - \theta_2) + (D_2^2h_2 - D_0^2h_0)\sin(\theta_1) \]

(A.4)

\[ K_1 = \frac{D_2^0}{D_1^2} \left[1 + k_0\cos(\theta_1) + h_0\sin(\theta_1) \right] \]

(A.5)
\[ K_2 = \left( \frac{D_0}{D_1} \right) \left[ D_0 - D_1 \right] \left[ h_0 \cos(\theta_1) - k_0 \sin(\theta_1) \right] \]  

(A.6)

\[ K_3 = 0 \]  

(A.7)

\[ L_1 = -\frac{\csc(\theta_1 - \theta_2)}{2D_1^2} \left[ 2 \left( D_1^2 + D_0^2 \right) - 2 \left( D_1^2 + D_2^2 \right) \cos(\theta_1 - \theta_2) - D_1^2 k_2 \cos(\theta_1 - 2\theta_2) + \left( 2D_1^2 k_0 - D_2^2 k_2 \right) \cos(\theta_1) + \ldots \right. \] 

\[ \left. + D_2^2 h_2 \sin(\theta_1 - 2\theta_2) + \left( 2D_0^2 h_0 - D_2^2 h_2 \right) \sin(\theta_1) \right] \]  

(A.8)

\[ L_2 = \frac{\csc(\theta_1 - \theta_2)}{D_1} \left[ \left( D_1^2 - D_2^2 \right) + \left( D_1^2 - D_0^2 \right) \cos(\theta_1 - \theta_2) + \left( D_2^2 k_2 - D_0^2 k_0 \right) \cos(\theta_2) + \left( D_2^2 h_2 - D_0^2 h_0 \right) \sin(\theta_2) \right] \]  

(A.9)

\[ L_3 = \frac{\csc(\theta_1 - \theta_2)}{4D_1} \left[ 4 \left( D_1^2 - D_2^2 \right) + 2 \left( D_1^2 k_0 + D_1^2 k_2 + 2D_2^2 k_2 \right) \cos(\theta_1 - \theta_2) + \left( 2D_1^2 k_0 - D_2^2 k_2 - D_2^2 k_0 \right) \cos(2\theta_1 - \theta_2) + \ldots \right. \] 

\[ \left. + \left( D_2^2 k_2 - D_1^2 k_2 \right) \cos(\theta_1 - 3\theta_2) + 4 \left( D_1^2 - D_2^2 \right) \cos(\theta_1 - \theta_2) + \left( 2D_0^2 h_0 + D_2^2 h_2 - 2D_2^2 h_2 \right) \sin(\theta_2) + \ldots \right. \] 

\[ \left. + \left( 2D_0^2 h_0 - D_2^2 h_2 - D_1^2 D_2 h_2 \right) \sin(\theta_1 - \theta_2) + \left( D_2^2 D_2 h_2 - D_2^2 h_2 \right) \sin(\theta_1 - 3\theta_2) \right] \]  

(A.10)

\[ M_1 = \left( \frac{D_2^2}{D_1^2} \right) \left[ 1 + k_2 \cos(\theta_2) + h_2 \sin(\theta_2) \right] \]  

(A.11)

\[ M_2 = 0 \]  

(A.12)

\[ M_3 = \left( \frac{D_2^2}{D_1^2} \right) \left[ D_1^2 - D_2^2 \right] \left[ h_2 \cos(\theta_2) - k_2 \sin(\theta_2) \right] \]  

(A.13)

REFERENCES

1. HOHMANN, W. "Die Erreichbarkeit der Himmelskorper", Oldenbourg, Munich, 1925.